

A MOTIVIC FORMULA FOR THE L -FUNCTION OF AN ABELIAN VARIETY OVER A FUNCTION FIELD

BRUNO KAHN

ABSTRACT. Let A be an abelian variety over the function field of a smooth projective curve C over an algebraically closed field k . We compute the l -adic cohomology groups

$$H^i(C, j_* H^1(\bar{A}, \mathbf{Q}_l)), \quad j : \eta \hookrightarrow C$$

in terms of arithmetico-geometric invariants of A . We apply this, when k is the algebraic closure of a finite field, to a motivic computation of the L -function of A .

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INTRODUCTION

Let K be a global field, and let A be an abelian variety over K . Its L -function is classically defined as

$$L(A, s) = \prod_{v \in \Sigma_K^f} \det(1 - \pi_v N(v)^{-s} \mid H_{l_v}^1(A)^{I_v})^{-1}$$

where Σ_K^f is the set of non-archimedean places of K , $H_{l_v}^1(A) = H_{\text{ét}}^1(\bar{A}, \mathbf{Q}_{l_v})$ is geometric l_v -adic cohomology of A (alternately, the dual of the Tate module $V_{l_v}(A)$) for some prime l_v different from the residue characteristic at finite v , I_v is the absolute inertia group at v and π_v is the geometric Frobenius at v , well-defined modulo I_v as a conjugacy class.

This function is independent of the choice of the l_v 's, as a consequence of Weil's Riemann hypothesis for curves and the “weight-monodromy conjecture”, which is known in this case by [SGA7, exp. IX, Th. 4.3 and Cor. 4.4]. In positive characteristic we have the more precise

Theorem 1. *Suppose $\text{char } K > 0$; let $k = \mathbf{F}_q$ be the field of constants of K . Then*

a) *(Grothendieck [GS], see also [Dc, §10].) One has the formula*

$$L(A, s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}$$

where $P_i \in \mathbf{Q}[t]$ with $P_i(0) = 1$. Moreover, $L(A, s)$ has a functional equation of the form

$$L(A, 2 - s) = ab^s L(A, s)$$

for suitable integers a, b .

b) *(Deligne [W.II].) The polynomials P_i have integer coefficients; the inverse roots of P_i are Weil q -numbers of weight $i + 1$.*

In this note, we give a formula for the polynomial P_i in terms of *pure motives over k* . To express the result, let us take some notation:

- $B = \text{Tr}_{K/k} A$ is the K/k -trace of A .
- $\text{LN}(A, K\bar{k}/\bar{k}) = A(K\bar{k})/B(\bar{k})$ is the *geometric Lang-Néron group* of A , where \bar{k} is an algebraic closure of k : it is finitely generated by the Lang-Néron theorem (e.g. [K1, App. B] or [C]). We view it as a Galois representation of k .

Theorem 2. *Let $\mathcal{M} = \mathcal{M}_{\text{rat}}(k, \mathbf{Q})$ be the category of pure motives over k with rational coefficients, modulo rational equivalence.¹*

a) *We have*

$$P_0(t) = Z(h^1(B), t), \quad P_2(t) = Z(h^1(B), qt)$$

where $h^1(B) \in \mathcal{M}$ is the degree 1 part of the Künneth decomposition of the motive of B .

b) *We have*

$$P_1(t) = Z(\text{ln}(A, K/k), qt)^{-1} \cdot Z(\text{III}(A, K/k), t)^{-1}$$

(a product of two polynomials), where $\text{ln}(A, K/k)$ is the Artin motive associated to $\text{LN}(A, K\bar{k}/\bar{k})$ and $\text{III}(A, K/k) \in \mathcal{M}$ is an effective Chow motive of weight 2 whose l -adic realization is $V_l(\text{III}(A, K\bar{k}))$, where $\text{III}(A, K\bar{k})$ is the geometric Tate-Šafarevič group of A .

¹Throughout this paper we use the contravariant convention for pure motives, e.g. as in [K1].

In Theorem 2, we used the Z -function of a motive $M \in \mathcal{M}$ [Kl]. It is known to be a rational function of t , with a functional equation; more precisely, if M is homogeneous of weight w , then $Z(M, t)$ is a polynomial or the inverse of a polynomial according as w is odd or even. That its inverse roots are Weil q -numbers of weight w depends on [W.I] rather than [W.II]. Theorem 2 also provides a proof that $L(A, s)$ is independent of the l_v 's avoiding [SGA7, Exp. IX].

The motive $\mathfrak{m}(A, K/k)$ is really the new character in this story. We construct it “by hand” in Proposition 4.3; however, we will show in [K3] that it is actually canonical and functorial in A (for homomorphisms of abelian varieties).

Theorem 2 “reduces” the Birch and Swinnerton-Dyer conjecture for A to the non-vanishing of $Z(\mathfrak{m}(A, K/k), t)$ at $t = q^{-1}$. The existence of $\mathfrak{m}(A, K/k)$ actually yields a simple proof of the following theorem of Kato and Trihan by basically quoting the relevant literature [I, M, IR]:

Corollary 1 ([KT]). *The following conditions are equivalent:*

- (i) $\text{ord}_{s=1} L(A, s) = \text{rk} A(K)$.
- (ii) $\text{III}(A, K)\{l\}$ is finite for some prime l .
- (iii) $\text{III}(A, K)\{l\}$ is finite for all primes l .
- (iv) $\text{III}(A, K)$ is finite.

(We almost don't touch the special value at $s = 1$, see however §6.3.)

To prove Theorem 2, we start from Grothendieck's formula for $P_i(t)$ (here we take $l_v = l \nmid q$ for all v):

$$(0.1) \quad P_i(t) = \det(1 - \pi_k t \mid H^i(\bar{C}, j_* H_l^1(A)))$$

where π_k is the geometric Frobenius of k , C is the smooth projective k -curve with function field K and $j : \text{Spec } K \hookrightarrow C$ is the inclusion of the generic point. The issue is then to give an expression of the cohomology groups $H^i(\bar{C}, j_* H_l^1(A))$: this is done in Theorem 1.1 below when A is the Jacobian of a curve, and in Corollary 1.2 in general.

When A is the Jacobian J of a curve Γ , we also get a precise relationship between $L(J, s)$ and the zeta function of a smooth projective k -surface spreading Γ , which was my original motivation for this work. More precisely, let Γ be a regular, projective, geometrically irreducible curve over K and S a smooth projective surface over k , fibred over C by a flat morphism f , with generic fibre Γ :

$$(0.2) \quad \begin{array}{ccc} \Gamma & \longrightarrow & S \\ f' \downarrow & & f \downarrow \\ \text{Spec } K & \xrightarrow{j} & C \\ & & p \downarrow \\ & & \text{Spec } k. \end{array}$$

Define (cf. [S])

$$L(h^i(\Gamma), s) = \prod_{v \in \Sigma_K} \det(1 - \pi_v N(v)^{-s} \mid H_l^i(\Gamma)^{I_v})^{(-1)^{i+1}}$$

$$L(h(\Gamma), s) = \prod_{i=0}^2 L(h^i(\Gamma), s)$$

so that

$$L(h^0(\Gamma), s) = \zeta(C, s), \quad L(h^2(\Gamma), s) = \zeta(C, s-1),$$

$$L(h^1(\Gamma), s) = L(J, s)^{-1}$$

(beware the exponent change!).

Theorem 3. *We have*

$$\frac{\zeta(S, s)}{L(K, h(\Gamma), s)} = Z(a(D), q^{1-s})$$

where $a(D)$ is the Artin motive associated to the “divisor of multiple fibres”

$$D = \bigoplus_{c \in C_{(0)}} \bar{D}_c, \quad \bar{D}_c = \text{Coker}(\mathbf{Z} \xrightarrow{f^*} \bigoplus_{x \in \text{Supp}(f^{-1}(c))} \mathbf{Z}).$$

Theorem 2, Corollary 1 and Theorem 3 are results on abelian varieties over a global field of positive characteristic. More intriguing for me is that Theorem 2 leads to a definition of the L -function of an abelian variety over a *finitely generated field of Kronecker dimension 2*, see Definition 7.1. This might be viewed a step towards answering the awkwardness of [T2, §4]: meanwhile, it raises more questions than it answers.

The main technical part of this work is to prove Theorem 1.1 below. The method is to “ l -adify” Grothendieck’s computations with \mathbb{G}_m coefficients in [Br.III, §4]². In a forthcoming work with Amílcar Pacheco [KP], we shall extend these results to a general fibration of smooth projective k -varieties, with a different and (hopefully) less unpleasant proof.

Contents of this paper. Theorem 1.1 and Corollary 1.2 are stated in Section 1. The first is proven in Section 2 and the second in Section 3: as explained above, Corollary 1.2 implies Theorem 2. In section 4, we show how Theorem 1.1 yields an identity in K_0 of a category of l -adic representations or pure motives, see Theorem 4.1: it implies

²These computations also appear with less generality in two other exposés of the volume *Dix exposés sur la cohomologie des schémas*: [Ra, §3] and [T1, Th. 3.1].

Theorem 3. In Section 5, we recall well-known facts on the crystalline realisation and present them in a convenient way. In Section 6, we examine what Theorem 2 teaches us on the functional equation and special values of $L(A, s)$; in particular, we prove Corollary 1 in §6.2. Finally, in Section 7, we get a formula for the total L -function of a surface S over a global field k in terms of L -functions of motives over k associated to a fibration of S over a curve (Theorem 7.4).

Acknowledgements. This work was partly inspired by the papers of Hindry-Pacheco [HP] and Hindry-Pacheco-Wazir [HPW]; I would also like to acknowledge several discussions with Amílcar Pacheco around it, which eventually led to [KP]. For this, I thank the Réseau franco-brésilien de mathématiques (RFBM) for its support for two visits to Rio de Janeiro in 2008 and 2010.

Theorems 2 (for the Jacobian of a curve), 3, 1.1 and 4.1 were obtained in the fall 2008 at the Tata Institute of Fundamental Research of Mumbai during its p -adic semester; I thank this institution for its hospitality and R. Sujatha for having invited me. These results were initially part of a more ambitious project on adjunctions in categories of motives [K3], from which I extracted them. The rest of the present article was obtained more recently.

1. COHOMOLOGICAL RESULTS

Consider the situation of (0.2), with k separably closed. Take a prime number l invertible in k . We write

$$H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) = \text{Coker}(\text{NS}(S) \otimes \mathbf{Q}_l \rightarrow H^2(S, \mathbf{Q}_l(1)))$$

where $\text{NS}(S)$ is the Néron-Severi group of S . Here are two other descriptions of this group:

$$(1.1) \quad H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) \simeq V_l(\text{Br}(S)) \quad (\text{Kummer exact sequence}).$$

$$(1.2) \quad H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) \simeq V_l(\text{III}(J, K)) \quad [\text{Br.III, pp. 120/121}]$$

where J is the Jacobian variety of Γ and $\text{III}(J, K)$ denotes its Tate-Šafarevič group.

Theorem 1.1. *Suppose k separably closed. There are isomorphisms*

$$\begin{aligned} H^0(C, j_* R^1 f'_* \mathbf{Q}_l(1)) &\simeq H^1(\text{Tr}_{K/k} J, \mathbf{Q}_l(1)) \\ H^2(C, j_* R^1 f'_* \mathbf{Q}_l(1)) &\simeq H^1(\text{Tr}_{K/k} J, \mathbf{Q}_l) \end{aligned}$$

and an exact sequence

$$0 \rightarrow \text{LN}(J, K/k) \otimes \mathbf{Q}_l \rightarrow H^1(C, j_* R^1 f'_* \mathbf{Q}_l(1)) \rightarrow H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) \rightarrow 0.$$

Corollary 1.2. *Let $f' : A \rightarrow \operatorname{Spec} K$ be an abelian variety over K . Suppose k separably closed. There are isomorphisms*

$$\begin{aligned} H^0(C, j_* R^1 f'_* \mathbf{Q}_l(1)) &\simeq H^1(\operatorname{Tr}_{K/k} A, \mathbf{Q}_l(1)) \\ H^2(C, j_* R^1 f'_* \mathbf{Q}_l(1)) &\simeq H^1(\operatorname{Tr}_{K/k} A, \mathbf{Q}_l) \end{aligned}$$

and an exact sequence

$$0 \rightarrow \operatorname{LN}(A, K/k) \otimes \mathbf{Q}_l \rightarrow H^1(C, j_* R^1 f'_* \mathbf{Q}_l(1)) \rightarrow V_l(\operatorname{III}(A, K)) \rightarrow 0.$$

If $k = \mathbf{C}$, it seems likely that similar results hold for the analytic cohomology of $R^1 f'_* \mathbf{Q}(1)$ with similar techniques as those used in the next section (replacing Kummer sequences by exponential sequences), but I haven't tried to prove them.

2. PROOF OF THEOREM 1.1

2.1. Reduction to the cohomology of the Néron model.

Lemma 2.1. *Let $\mathcal{J} = j_* J$ be the Néron model of J over C . There are short exact sequences*

$$\begin{aligned} 0 \rightarrow (\varprojlim H^{p-1}(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} &\rightarrow H^p(C, j_* R^1 f'_* \mathbf{Q}_l(1)) \\ &\rightarrow V_l(H^p(C, \mathcal{J})) \rightarrow 0. \end{aligned}$$

Proof. Given $c \in C$, write $i_c : c \hookrightarrow C$ for the corresponding closed immersion and let Φ_c be the group of connected components of the special fibre of \mathcal{J} at c . Then Φ_c is finite for any c and is 0 except for a finite number of c 's. Write $\mathcal{J}^0 = \operatorname{Ker}(\mathcal{J} \rightarrow \bigoplus_{c \in C} (i_c)_* \Phi_c)$ for the connected component of \mathcal{J} . Since k is separably closed, we have isomorphisms $H^p(C, \mathcal{J}^0) \xrightarrow{\sim} H^p(C, \mathcal{J})$ for $p > 0$ and an injection with finite cokernel $H^0(C, \mathcal{J}^0) \hookrightarrow H^0(C, \mathcal{J})$. So,

$$\begin{aligned} (\varprojlim H^*(C, \mathcal{J}^0)/l^\nu) \otimes \mathbf{Q} &\xrightarrow{\sim} (\varprojlim H^*(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q}, \\ V_l(H^*(C, \mathcal{J}^0)) &\xrightarrow{\sim} V_l(H^*(C, \mathcal{J})). \end{aligned}$$

To handle the cohomology of \mathcal{J}^0 , we may use the Kummer exact sequences

$$0 \rightarrow {}_{l^\nu} \mathcal{J}^0 \rightarrow \mathcal{J}^0 \xrightarrow{l^n} \mathcal{J}^0 \rightarrow 0$$

which yield exact sequences with finite central terms

$$0 \rightarrow H^{p-1}(C, \mathcal{J}^0)/l^\nu \rightarrow H^p(C, {}_{l^\nu} \mathcal{J}^0) \rightarrow {}_{l^\nu} H^p(C, \mathcal{J}^0) \rightarrow 0$$

hence other exact sequences

$$0 \rightarrow \varprojlim H^{p-1}(C, \mathcal{J}^0)/l^\nu \otimes \mathbf{Q} \rightarrow H^p(C, V_l(\mathcal{J}^0)) \rightarrow V_l(H^p(C, \mathcal{J}^0)) \rightarrow 0.$$

But $V_l(\mathcal{J}^0) \xrightarrow{\sim} V_l(\mathcal{J})$; as $R^1 f'_* \mu_{l^\nu} \xrightarrow{\sim} {}_{l^\nu} J$ and $j_* {}_{l^\nu} J = {}_{l^\nu} \mathcal{J}$, the lemma follows. \square

2.2. Cohomology of $B := j_* \text{Pic}_{\Gamma/K}$. For $c \in C$, view $f^{-1}(c)$ as a divisor on S . Let $D_c = \bigoplus_{x \in \text{Supp}(f^{-1}(c))} \mathbf{Z}$ and $\bar{D}_c = D_c / \langle f^{-1}(c) \rangle$: thus $\bar{D}_c = 0$ whenever f is smooth over c . Write

$$(2.1) \quad D = \bigoplus_{c \in C} \bar{D}_c$$

(a finite sum).

Lemma 2.2. *There is an isogeny*

$$\text{Pic}_{S/k}^0 / \text{Pic}_{C/k}^0 \rightarrow \text{Tr}_{K/k} J$$

and a complex

$$0 \rightarrow \text{NS}(C) \rightarrow \text{NS}(S) \rightarrow \text{Pic}(\Gamma) / \text{Tr}_{K/k} J(k) \rightarrow 0$$

which, modulo finite groups, is acyclic except at $\text{NS}(S)$, where its homology is D .

Proof. This follows from [HP, prop. 3.3 et 3.8] or [K2, 3.2 a)]. \square

Lemma 2.3. *a) There is an exact sequence*

$$(2.2) \quad 0 \rightarrow D \rightarrow \text{Pic}(S/C) \rightarrow H^0(C, B) \rightarrow 0$$

where $\text{Pic}(S/C) = H^0(C, \text{Pic}_{S/C})$ and $B = j_* \text{Pic}_{\Gamma/K}$.

b) There is an exact sequence

$$(2.3) \quad 0 \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(S/C) \rightarrow 0$$

and isomorphisms

$$(2.4) \quad H^n(S, \mathbb{G}_m) \xrightarrow{\sim} H^{n-1}(C, B) \text{ for } n > 1.$$

In particular,

$$(2.5) \quad \text{Br}(S) \xrightarrow{\sim} H^1(C, B)$$

and $H^n(C, B) = 0$ for $n > 3$.

Proof. This follows from the computations in [Br.III, §4]. After [Br.III, (4.1)], we have a long cohomology exact sequence (a consequence of *op. cit.*, (3.2)):

$$\dots \rightarrow H^n(C, \mathbb{G}_m) \rightarrow H^n(S, \mathbb{G}_m) \rightarrow H^{n-1}(C, P) \rightarrow \dots$$

where $P = \text{Pic}_{S/C}$. Moreover, the homomorphism $P \rightarrow B$ is epi and its kernel is a skyscraper sheaf whose global sections are D [Br.III, p. 114]: (2.2) follows, as well as isomorphisms $H^n(C, P) \xrightarrow{\sim} H^n(C, B)$ for $n > 0$. As $H^n(C, \mathbb{G}_m) = 0$ for $n > 1$ and $H^n(S, \mathbb{G}_m) = 0$ for $n > 4$, one gets (2.3) and (2.4). \square

2.3. l -adic conversion. Using (2.3), (2.4) and the structure of $\text{Pic}(C)$ and $\text{Pic}(S)$, we find exact sequences

$$\begin{aligned} 0 \rightarrow V_l(\text{Pic}^0(C)) \rightarrow V_l(\text{Pic}^0(S)) \rightarrow V_l(\text{Pic}(S/C)) \\ \rightarrow \mathbf{Q}_l \rightarrow \text{NS}(S) \otimes \mathbf{Q}_l \rightarrow (\varprojlim \text{Pic}(S/C)/l^\nu) \otimes \mathbf{Q} \rightarrow 0 \end{aligned}$$

$$\begin{aligned} 0 \rightarrow V_l(\text{Pic}(S/C)) \rightarrow V_l(H^0(C, B)) \rightarrow D \otimes \mathbf{Q}_l \\ \rightarrow (\varprojlim \text{Pic}(S/C)/l^\nu) \otimes \mathbf{Q} \rightarrow (\varprojlim H^0(C, B)/l^\nu) \otimes \mathbf{Q} \rightarrow 0 \\ H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) \xrightarrow{\sim} V_l(H^1(C, B)), \quad (\varprojlim H^1(C, B)/l^\nu) \otimes \mathbf{Q} = 0. \end{aligned}$$

Using Lemma 2.2, we derive new exact sequences

$$\begin{aligned} 0 \rightarrow V_l(\text{Pic}^0(C)) \rightarrow V_l(\text{Pic}^0(S)) \rightarrow V_l(\text{Tr}_{K/k} J) \rightarrow 0 \\ 0 \rightarrow D \otimes \mathbf{Q}_l \rightarrow (\text{NS}(S)/\mathbf{Z}) \otimes \mathbf{Q}_l \rightarrow (\text{Pic}(\Gamma)/(\text{Tr}_{K/k} J)(k)) \otimes \mathbf{Q}_l \rightarrow 0. \end{aligned}$$

Hence

$$\begin{aligned} V_l(\text{Tr}_{K/k} J) \xrightarrow{\sim} V_l(\text{Pic}(S/C)) \\ (\text{NS}(S)/\mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} (\varprojlim \text{Pic}(S/C)/l^\nu) \otimes \mathbf{Q} \end{aligned}$$

then

$$\begin{aligned} V_l(\text{Tr}_{K/k} J) \xrightarrow{\sim} V_l(H^0(C, B)) \\ (\text{Pic}(\Gamma)/(\text{Tr}_{K/k} J)(k)) \otimes \mathbf{Q}_l \xrightarrow{\sim} (\varprojlim H^0(C, B)/l^\nu) \otimes \mathbf{Q}. \end{aligned}$$

2.4. From B to \mathcal{J} . To pass from B to \mathcal{J} , we work in the category \mathcal{C} of abelian groups modulo the thick sub-category of finite groups, which does not affect the functor V_l .

Lemma 2.4. *In \mathcal{C} , we have*

- (1) *A split exact sequence $0 \rightarrow H^0(C, \mathcal{J}) \rightarrow H^0(C, B) \rightarrow \mathbf{Z} \rightarrow 0$.*
- (2) *An isomorphism $H^1(C, \mathcal{J}) \simeq \text{Br}(S)$.*
- (3) *An isomorphism $H^2(C, \mathcal{J})\{l\} \simeq \text{Im}_{K/k} J\{l\}(-1)$; $H^2(C, \mathcal{J})$ is torsion.*
- (4) *$H^p(C, \mathcal{J}) = 0$ for $p \geq 3$.*

Proof. We have an exact sequence, split in \mathcal{C}

$$(2.6) \quad 0 \rightarrow \mathcal{J} \rightarrow B \rightarrow \mathbf{Z} \rightarrow 0$$

which gives (still in \mathcal{C}) split exact sequences

$$0 \rightarrow H^p(C, \mathcal{J}) \rightarrow H^p(C, B) \rightarrow H^p(C, \mathbf{Z}) \rightarrow 0.$$

For $p = 0$, we get (1). For $p = 1$, we get (2) in view of $H^1(C, \mathbf{Z}) = 0$ and (2.5).

For $p > 1$, this gives in view of (2.4)

$$0 \rightarrow H^p(C, \mathcal{J}) \rightarrow H^{p+1}(S, \mathbb{G}_m) \xrightarrow{\tau} H^p(C, \mathbf{Z}) \rightarrow 0$$

Still for $p > 1$, we have isomorphisms

$$H^{p-1}(C, \mathbf{Q}_l/\mathbf{Z}_l) \xrightarrow{\sim} H^p(C, \mathbf{Z})\{l\}, \quad H^{p+1}(S, \mathbf{Q}_l/\mathbf{Z}_l(1)) \xrightarrow{\sim} H^{p+1}(S, \mathbb{G}_m)\{l\}.$$

The morphism τ then gets identified to the trace morphism, which is an isomorphism for $p \geq 3$ (hence (4)), while for $p = 2$ it is the dual of

$$f^* : H^1(C, \mathbf{Z}_l(1)) \rightarrow H^1(S, \mathbf{Z}_l(1)).$$

We can then identify $H^2(C, \mathcal{J})$ with $\text{Ker}(\text{Alb}(S)\{l\}(-1) \rightarrow \text{Alb}(C)\{l\}(-1))$, i.e., to $\text{Im}_{K/k} J\{l\}(-1)$. \square

2.5. Conclusion. From Lemma 2.4 and the computations in §2.3, we derive

$$\begin{aligned} V_l(\text{Tr}_{K/k} J) &\xrightarrow{\sim} V_l(H^0(C, \mathcal{J})) \\ V_l(H^1(C, \mathcal{J})) &\simeq V_l(\text{Br}(S)) \simeq H_{\text{tr}}^2(S, \mathbf{Q}_l(1)). \\ \text{LN}(J, K/k) \otimes \mathbf{Q}_l &\xrightarrow{\sim} (\varprojlim H^0(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} \\ H_{\text{tr}}^2(S, \mathbf{Q}_l(1)) &\xrightarrow{\sim} V_l(H^1(C, \mathcal{J})), \quad (\varprojlim H^1(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} = 0 \\ V_l(H^2(C, \mathcal{J})) &\simeq V_l(\text{Im}_{K/k} J)(-1) \simeq V_l(\text{Tr}_{K/k} J)(-1) \\ (\varprojlim H^2(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} &= 0 \\ V_l(H^p(C, \mathcal{J})) &= (\varprojlim H^p(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} = 0 \text{ for } p > 2 \end{aligned}$$

and finally the isomorphisms and the exact sequence of Theorem 1.1, using Lemma 2.1 and the isomorphisms

$$V_l(A)(-1) \simeq V_l(A)^* \simeq H^1(A, \mathbf{Q}_l)$$

valid for any abelian variety A over a separably closed field. \square

3. PROOFS OF COROLLARY 1.2 AND THEOREM 2

If X is a smooth projective variety of dimension d over a field F , we write $CH_{\equiv}^d(X \times_F X)$ for the quotient of the ring of Chow correspondences on X by the ideal generated by those $Z \subset X \times X$ such that $p_1(Z) \neq X$ or $p_2(Z) \neq X$, where p_1, p_2 are the two projections $X \times X \rightarrow X$ (cf. [F, ex. 16.1.2 (b)].)

Proposition 3.1. *a) In the situation of (0.2), there is a ring isomorphism $CH_{\equiv}^1(\Gamma \times_K \Gamma) \xrightarrow{\sim} \text{End}_K(J)$, and a ring homomorphism*

$$r : CH_{\equiv}^1(\Gamma \times_K \Gamma) \rightarrow CH_{\equiv}^2(S \times_k S).$$

b) The rings $\text{End}_K(J) \otimes \mathbf{Q}$ and $CH_{\equiv}^2(S \times_k S) \otimes \mathbf{Q}$ act compatibly on the isomorphisms and the exact sequence of Theorem 1.1, as well as on (1.1) and (1.2).

Proof. a) The first isomorphism is due to Weil [W, ch. 6, th. 22]³. We have a homomorphism

$$R : Z^1(C \times_K C) \rightarrow Z^2(S \times_k S)$$

defined as follows: let $Z \subset \Gamma \times_K \Gamma$ be an irreducible cycle of codimension 1. Write \mathcal{Z} for its closure in $S \times_C S$. We set $R(Z) = \text{image of } \mathcal{Z} \text{ in } Z^2(S \times_k S)$. One checks that R passes to rational equivalence and to the equivalences \equiv , and that the induced map r is compatible with composition of correspondances.

A more functorial construction of r will be given in [K3].

b) This is a long but eventless verification. \square

In view of Theorem 1, Theorem 2 immediately follows from (0.1) and Corollary 1.2.

4. COMPARING CLASSES IN K_0 ; PROOF OF THEOREM 3

In (0.2), let us come back to the case of an arbitrary base field k . Let k_s be a separable closure of k and $G = \text{Gal}(k_s/k)$. Write $C_s, \Gamma_s \dots$ for the objects of (0.2) after base change to k_s . Then Theorem 1.1 “over k_s ” is G -equivariant; moreover, the Kk_s/k_s -trace of J_s is $(\text{Tr}_{K/k} J)_s$ ([K1, Prop. 6] or [C, Th. 6.8]). One might want to compare

$$R(pf)_* \mathbf{Q}_l$$

and

$$Rp_* j_* Rf'_* \mathbf{Q}_l$$

in the derived category of $\mathbf{Q}_l[[G]]$ -modules. Unfortunately this has no meaning, because j_* has no meaning in the derived category.

On the other hand, let \mathbf{K}_l be the Grothendieck group of continuous, finite dimensional \mathbf{Q}_l -representations G . We may consider in \mathbf{K}_l :

- (1) $H_l = [H_l^*(S_s)]$, the alternating sum of the l -adic cohomology groups of S ;
- (2) $H'_l = [R^* p_* j_* R^* f'_* \mathbf{Q}_l]$ (9 terms).

We may also consider D as a discrete topological G -module.

Theorem 4.1. $H_l - H'_l = [D \otimes \mathbf{Q}_l(-1)]$.

Proof. For simplicity, put $B := \text{Tr}_{K/k} J$. Let \mathbf{K} be the Grothendieck group of the category $\mathbf{Mot}_{\text{rat}}(k, \mathbf{Q})$ of Chow motives over k , and let $R_l : \mathbf{K} \rightarrow \mathbf{K}_l$ be the homomorphism given by l -adic realisation. (To avoid confusion, we adopt cohomological notation as in [K1] but contrary to [KMP], also for Tate twists.) Then $H_l = R_l(h)$, with $h = h(S)$.

³The homomorphism is constructed in [F, ex. 16.1.2 (c)], but its bijectivity is not mentioned. . .

Similarly, there exists a canonical $h' \in \mathbf{K}$ such that $H'_l = R_l(h')$. Indeed, Theorem 1.1 shows that

$$\begin{aligned} [p_* j_* R^1 f'_* \mathbf{Q}_l] &= R_l([h^1(B)]), [R^2 p_* j_* R^1 f'_* \mathbf{Q}_l] = R_l([h^1(B(-1))]), \\ [R^1 p_* j_* R^1 f'_* \mathbf{Q}_l] &= R_l([t^2(S)] + [\ln(J, K/k)(-1)]) \end{aligned}$$

where $\ln(J, K/k)$ is the Artin motive associated to the Galois module $\mathrm{LN}(J, Kk_s/k_s)$ (see [KMP] for $t^2(S)$).

Using $f'_* \mathbf{Q}_l = \mathbf{Q}_l$ and $R^2 f'_* \mathbf{Q}_l = \mathbf{Q}_l(-1)$, we similarly get

$$[R^q p_* j_* f'_* \mathbf{Q}_l] = R_l([h^q(C)]), \quad [R^q p_* j_* R^2 f'_* \mathbf{Q}_l] = R_l([h^q(C)(-1)]).$$

Set

$$\begin{aligned} h' &= \sum_{q=0}^2 (-1)^q [h^q(C)] + \sum_{q=0}^2 (-1)^q [h^q(C)(-1)] \\ &\quad - ([h^1(B)] + [h^1(B(-1))] - ([t^2(S)] + [\ln(J, K/k)(-1)])) \in \mathbf{K}. \end{aligned}$$

To prove Theorem 4.1, it therefore suffices to show:

$$(4.1) \quad h - h' = [D(-1)].$$

From Lemma 2.2, one gets identities in \mathbf{K} :

$$\begin{aligned} [h^1(S)] &= [h^1(C)] + [h^1(B)], [h^3(S)] = [h^1(C)(-1)] + [h^1(B(-1))], \\ [\mathrm{NS}(S)] &= [\mathrm{NS}(C)] + [\mathbf{1}] + [\ln(J, K/k)] + [D] = 2[\mathbf{1}] + [\ln(J, K/k)] + [D] \end{aligned}$$

from which (4.1) easily follows. \square

Remark 4.2. The same proof gives a more precise identity in \mathbf{K} :

$$[h^2(S)] - [Rp_* j_* h^1(J)] = [D(-1)] + 2[\mathbf{L}] - [h^1(B)] - [h^1(B)(-1)]$$

where \mathbf{L} is the Lefschetz motive and $[Rp_* j_* h^1(J)]$ stands for the canonical element of \mathbf{K} whose l -adic realisation is $\sum_{i=0}^2 (-1)^i [R^i p_* j_* f'_* \mathbf{Q}_l]$.

Let us pass to the case of an abelian variety A over K . We would like to interpret the terms in Corollary 1.2 as realisations of pure motives over k . This is clearly possible, except perhaps for the term $V_l(\mathrm{III}(A, Kk_s))$.

Proposition 4.3. *There exists an effective k -Chow motive $\mathfrak{m}(A, K/k)$ such that $R_l(\mathfrak{m}(A, K/k)) = V_l(\mathrm{III}(A, Kk_s))(-1)$.*

Proof. Write A as a direct summand of a Jacobian J , up to isogeny, where J comes from a situation (0.2). Via Proposition 3.1, the corresponding projector $\pi \in \mathrm{End}(A) \otimes \mathbf{Q}$ defines a projector $r(\pi) \in CH_{\equiv}^2(S \times_k S) \otimes \mathbf{Q} = \mathrm{End}(t^2(S))$ [KMP]. Define $\mathfrak{m}(A, K/k)$ as the image of $r(\pi)$. \square

Remark 4.4. We shall see in [K3] that the motive $\mathfrak{m}(A, K/k)$ is independent of the choice of J , and is *functorial in A* .

Theorem 3 immediately follows from Theorem 4.1.

5. THE CRYSTALLINE REALISATION

In this section, k is any perfect field of characteristic $p > 0$.

5.1. Isocrystals. We rely here on the crystal-clear exposition of Saavedra [Saa, Ch. VI, §3].

Let $W(k)$ be the ring of Witt vectors on k and $K(k)$ be the field of fractions of $W(k)$. The Frobenius automorphism $x \mapsto x^p$ of k lifts to an endomorphism on $W(k)$ and an automorphism of $K(k)$, written σ : we have $K(k)^\sigma = \mathbf{Q}_p$. A k -isocrystal is a finite-dimensional $K(k)$ -vector space M provided with a σ -linear automorphism F_M . k -isocrystals form a \mathbf{Q}_p -linear tannakian category $\mathbf{Fcriso}(k)$, provided with a canonical $K(k)$ -valued fibre functor (forgetting F_M) [Saa, VI.3.2.1]. We have

$$\mathbf{Fcriso}(k)(\mathbf{1}, M) = M^{F_M} = \{m \in M \mid F_M m = m\}$$

for $M \in \mathbf{Fcriso}(k)$, where $\mathbf{1} = (K(k), \sigma)$ is the unit object. For $n \in \mathbf{Z}$, we write more generally

$$(5.1) \quad M^{(n)} = M^{F_M=p^n} = \mathbf{Fcriso}(k)(\mathbf{L}_{\text{crys}}^n, M) = \mathbf{Fcriso}(k)(\mathbf{1}, M(n))$$

where $M(n) = M \otimes \mathbf{L}_{\text{crys}}^{-n}$ with $\mathbf{L}_{\text{crys}} := (K(k), p\sigma)$.

5.2. The realisation. By [Saa, VI.4.1.4.3], the formal properties of crystalline cohomology yield a \otimes -functor

$$R_p : \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathbf{Fcriso}(k).$$

This functor sends the motive of a smooth projective variety X to $H_{\text{crys}}^*(X/W(k)) \otimes_{W(k)} K(k)$ and the Lefschetz motive \mathbf{L} to \mathbf{L}_{crys} .

5.3. The case of a finite field. Suppose that $k = \mathbf{F}_q$, with $q = p^m$. Then any object $M \in \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q})$ has its *Frobenius endomorphism* π_M : if $M = h(X)$ for a smooth projective variety X , $\pi_M = \pi_X$ is the graph of the Frobenius endomorphism F^m on X . This implies:

Lemma 5.1. *The action of π_M on $R_p(M)$ equals that of F^m .* \square

Let \bar{k} be an algebraic closure of k . There is an obvious functor

$$(5.2) \quad \mathbf{Fcriso}(k) \rightarrow \mathbf{Fcriso}(\bar{k}), \quad M \mapsto \bar{M} := M \otimes_{K(k)} K(\bar{k})$$

which is compatible with the extension of scalars $\mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathbf{Mot}_{\text{rat}}(\bar{k}, \mathbf{Q})$ via the realisation functors R_p for k and \bar{k} . Moreover F^m is $K(k)$ -linear, therefore one can talk of its eigenvalues. We have the following result of Milne [M, Lemma 5.1]:

Lemma 5.2. *One has an equality*

$$\det(1 - \gamma t \mid \bar{M}^{(n)}) = \prod_{v(a)=v(q^n)} (1 - (q^n/at))$$

where γ is the arithmetic Frobenius and a runs through the eigenvalues of F^m having same valuation as q^n .

5.4. Logarithmic de Rham-Witt cohomology.

Proposition 5.3. *Let X/k be smooth projective. Then, for any $i, n \in \mathbf{Z}$, there is a canonical isomorphism*

$$H^i(X, \mathbf{Q}_p(n)) \xrightarrow{\sim} (H_{\text{crys}}^i(X/W(k)) \otimes_{W(k)} K(k))^{(n)}$$

where the left hand side is logarithmic Hodge-Witt cohomology as in Milne [M, p. 309].

Proof. This is [M, Prop. 1.15], but unfortunately its proof is garbled (the last line of loc. cit., p. 310 is wrong). Let us recapitulate it. For simplicity, let $W = W(k)$ and $K = K(k)$.

1) The slope spectral sequence

$$E_1^{i,j} = H^j(X, W\Omega^i) \Rightarrow H^{i+j}(X, W\Omega) \simeq H_{\text{crys}}^{i+j}(X/W)$$

degenerates up to torsion, yielding canonical isomorphisms of k -isocrystals

$$H^{i-n}(X, W\Omega^n) \otimes_W K \xrightarrow{\sim} (H^i(X/W) \otimes_W K)_{[n, n+1[}$$

where the index $[n, n+1[$ means the sum of summands of slope λ for $n \leq \lambda < n+1$ [I, Th. 3.2 p. 615 and (3.5.4) p. 616].

2) If k is algebraically closed, the homomorphism

$$H^i(X, \mathbf{Z}_p(n)) := H^{i-n}(X, W\Omega_{\log}^n) \rightarrow H^{i-n}(X, W\Omega^n)^F$$

is bijective [IR, Cor. 3.5 p. 194].

3) In general, descend from \bar{k} to k by taking Galois invariants. \square

By [M, §2] and [G], Chow correspondences act on logarithmic Hodge-Witt cohomology by respecting the isomorphisms of Proposition 5.3: this yields functors

$$H^i(-, \mathbf{Q}_p(n)) : \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathbf{Vec}_{\mathbf{Q}_p}^*$$

and natural isomorphisms

$$(5.3) \quad H^i(M, \mathbf{Q}_p(n)) \xrightarrow{\sim} \mathbf{Fcriso}(k)(L_{\text{crys}}^n, R_p(M)), \quad M \in \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}).$$

5.5. The Brauer group and the Tate-Šafarevič group. We have

Proposition 5.4 ([I, (5.8.5) p. 629]). *Let k be algebraically closed and X/k be smooth projective. Then there is an exact sequence*

$$0 \rightarrow \mathrm{NS}(X) \otimes \mathbf{Z}_p \rightarrow H^2(X, \mathbf{Z}_p(1)) \rightarrow T_p(\mathrm{Br}(X)) \rightarrow 0.$$

As before, Chow correspondences act on this exact sequence. Therefore if $X = S$ is a surface, applying the projector π_{tr}^2 defining $t^2(S)$, we get an isomorphism

$$H^2(t^2(X), \mathbf{Q}_p(1)) \simeq V_p(\mathrm{Br}(X))$$

hence, taking (5.3) into account:

$$\mathbf{Fcriso}(k)(\mathbf{L}_{\mathrm{crys}}, R_p(t^2(X))) \simeq V_p(\mathrm{Br}(X)).$$

If now K/k is a function field in one variable and A is an abelian variety over K , using the projector $r(\pi)$ from the proof of Proposition 4.3, we get an isomorphism

$$(5.4) \quad \mathbf{Fcriso}(k)(\mathbf{L}_{\mathrm{crys}}, R_p(\mathfrak{M}(A, K/k))) \simeq V_p(\mathbf{III}(A, K)).$$

6. FUNCTIONAL EQUATION, ORDER OF ZERO AND SPECIAL VALUE

6.1. Functional equation. Recall the functional equation of the zeta function of a pure motive M of weight w over a finite field k with q elements:

$$\zeta(M^*, -s) = \det(M)(-q^{-s})^{\chi(M)} \zeta(M, s)$$

where M^* is the dual of M , $\chi(M)$ is the Euler characteristic of M (computed for example with the help of its l -adic realisation) and

$$\det(M) = \pm q^{w\chi(M)/2}$$

is the determinant of the Frobenius endomorphism of M . Applying this to Theorem 2, we get the following functional equation for $L(K, A, s)$:

$$L(K, A, 2-s) = a(-q^{-s})^\beta L(K, A, s)$$

with

$$\begin{aligned} \beta &= -2\chi(h^1(B)) - \chi(\mathfrak{M}(A, K/k)) - \chi(\mathrm{In}(A, K/k)) \\ &= 4 \dim B - \mathrm{cork} \mathbf{III}(A, K\bar{k}) - \mathrm{rk} A(K\bar{k}) \\ a &= (\det h^1(B) \det h^1(B)(-1) \det \mathfrak{M}(A, K/k) \det \mathrm{In}(A, K/k)(-1))^{-1} \\ &= \pm q^\beta. \end{aligned}$$

The exponent β compares mysteriously with the one appearing in the functional equation of Grothendieck:

$$\beta = \chi(j_* H^1(\bar{A}, \mathbf{Q}_l)) = 2 - 2g - \deg(\mathfrak{f})$$

where g is the genus of C and \mathfrak{f} is the conductor of A (relative to K/k), and the second equality follows from [Ra, Th. 1].

6.2. Order of zero. From Theorem 2, one immediately gets the well-known equality and inequality

$$(6.1) \quad \begin{aligned} \text{ord}_{s=1} L(A, s) &= \text{rk} A(K) + \text{cork}_l^1 l\text{III}(A, K\bar{k}) \\ &\geq \text{rk} A(K) + \text{cork}_l l\text{III}(A, K) \end{aligned}$$

where $\text{cork}_l^1 l\text{III}(A, K\bar{k})$ is the corank of the generalised eigensubgroup for the eigenvalue 1 of the action of the arithmetic Frobenius γ on an arbitrary l -primary component of $\text{III}(A, K\bar{k})$ (cf. [Sch, Lemma 2 (i)] for $l \neq p$).

Indeed, let us show that $\text{ord}_{s=1} Z(\mathfrak{m}(A, K/k)) = \text{cork}_l^1 l\text{III}(A, K\bar{k})$ for any prime l . This order can be computed through the action of the Frobenius endomorphism $\pi_{\mathfrak{m}}$ of $\mathfrak{m}(A, K/k)(1)$ on $R(\mathfrak{m}(A, K/k)(1)) = R_l(\mathfrak{m}(A, K/k))(-1)$ for any realisation functor R on $\mathbf{Mot}_{\text{rat}}(k, \mathbf{Q})$. If we use the l -adic realisation R_l for $l \neq p$, the claim is clear by Proposition 4.3 since $R_l(\pi_{\mathfrak{m}})$ acts like the inverse of γ . If we now take R_p , we find from (5.1), (5.2) and (5.4):

$$R_p(\mathfrak{m}(A, K\bar{k}/\bar{k})(1))^{(0)} \simeq \overline{R_p(\mathfrak{m}(A, K/k)(1))}^{(0)} \simeq V_p(\text{III}(A, K\bar{k})).$$

By Lemma 5.2, we have

$$\det(1 - \gamma t \mid \overline{R_p(\mathfrak{m}(A, K/k)(1))}^{(0)}) = \prod_{v(a)=0} (1 - 1/at)$$

where a runs through the eigenvalues of F^m acting on $R_p(\mathfrak{m}(A, K/k)(1))$, with valuation 0. By lemma 5.1, $F^m = R_p(\pi_{\mathfrak{m}})$, so we are done.

This argument does not show that $\text{cork}_l l\text{III}(A, K) = \text{cork}_l l\text{III}(A, K\bar{k})^\gamma$ is independent of l . However, it does yield:

Proof of Corollary 1. For any l , $\text{cork}_l l\text{III}(A, K) = 0 \iff \text{III}(A, K)\{l\}$ is finite. In view of (6.1), this shows (i) \iff (ii) \iff (iii). To get (iii) \Rightarrow (iv), consider a surface S/k used to construct the motive $\mathfrak{m}(A, K/k)$. The projector $r(\pi)$ in the proof of Proposition 4.3 is represented by an algebraic correspondence with \mathbf{Q} coefficients, which have a common denominator D . For l prime to D , $\text{III}(A, K)\{l\}$ is then a direct summand of $\text{Br}(S)\{l\}$, a group of cofinite type whose finite quotient is 0 for almost all l . \square

6.3. Special value. It is less obvious to relate Theorem 2 to the value of the principal part in the Birch and Swinnerton-Dyer conjecture ([Sch, Theorem p. 509], [KT]):

$$(6.2) \quad \lim_{s \rightarrow 1} \frac{L(A, s)}{(s-1)^\rho} \sim \pm q^\rho \frac{|\text{III}(A, K)| |\det \langle, \rangle_{A(K)}|}{|A(K)_{\text{tors}}| |A'(K)_{\text{tors}}|} \prod_{c \in C} |\Phi_c(k(c))|,$$

where $\rho = \text{rk} A(K)$, $\langle, \rangle_{A(K)}$ is the height pairing constructed in [Sch, p. 502] and the Φ_c are the groups of connected components of the Néron model of A over C , as in §2.1.

It seems that the explicit expression of $L(A, s)$ could actually be used to provide an expression of the left hand side of (6.2) independently of the Birch and Swinnerton-Dyer conjecture, in the spirit of (6.1). This can presumably be done by the method of [Sch]: I did not succeed and leave it to better experts. Let me only note that in Theorem 2, the factors $Z(h^1(B), q^{-s})$ and $Z(h^1(B), q^{1-s})$ respectively contribute by $|B'(k)|$ and $|B(k)|$ (as usual, $B := \text{Tr}_{K/k} A$), while $Z(\ln(A, K/k), q^{1-s})$ contributes by

$$\pm q^{-\text{rk} A(K)} \frac{\det \langle, \rangle_{A(K\bar{k})}}{\det \langle, \rangle_{A(K)}} \frac{|A(K)_{\text{tors}}/B(k)|}{|(\text{LN}(A, K\bar{k}/\bar{k})_F)_{\text{tors}}|}$$

Where $\langle, \rangle_{A(K)}$ and $\langle, \rangle_{A(K\bar{k})}$ are the height pairings constructed in [Sch, p. 502]. This follows from the elementary lemma, in the spirit of [T1, Lemma z.4]:

Lemma 6.1. *Let $\langle, \rangle : M \times M' \rightarrow \mathbf{Q}$ be a \mathbf{Q} -non-degenerate pairing between finitely generated abelian groups. Suppose M and M' are provided with operators F, F' which are adjoint for the pairing, and \mathbf{Q} -semi-simple (e.g., F is of finite order). Let $P = \det(1 - FT)$ be the inverse characteristic polynomial of F acting on $M_{\mathbf{Q}}$. Then $\rho := \text{ord}_{T=1} P = \text{rk} M^F$ and, if $P' = P/(1 - T)^\rho$,*

$$|P'(1)| = \frac{\det \langle, \rangle^F}{\det \langle, \rangle} \frac{|(M^F)_{\text{tors}}|}{|(M_F)_{\text{tors}}|}$$

where M^F (resp. M_F) denotes the F -invariants (resp. coinvariants) of F and \langle, \rangle^F is the (\mathbf{Q} -non-degenerate) pairing induced by \langle, \rangle on $M^F \times M'^{F'}$.

7. SURFACES OVER A GLOBAL FIELD

In §4, suppose k global: K is a function field in one variable over k .

Definition 7.1. If A is an abelian variety over K , we set

$$L(K, h^1(A), s) = L(k, h^1(\mathrm{Tr}_{K/k} A), s) L(k, h^1(\mathrm{Tr}_{K/k} A), s - 1) \\ L(k, \mathfrak{m}(A, K/k), s) L(k, \ln(A, K/k), s - 1)$$

where the right hand side is defined in terms of l -adic realisations.

In the right hand side, the motive $\ln(A, K/k)$ is of weight 0 (it is an Artin motive), $h^1(\mathrm{Tr}_{K/k} A)$ is of weight 1 and $\mathfrak{m}(A, K/k)$ is of weight 2, a direct summand of h^2 of a suitable surface. Definition 7.1 is independent of the choice of l (invertible in k) because this is so for each individual factor (for $\mathfrak{m}(A, K/k)$, it follows from [RZ, Satz 2.13] and [Sa, cor. 0.6]).

If k is a number field, it may always be chosen as the algebraic closure of \mathbf{Q} in K , and this choice is unique. On the other hand, I don't know the answer to:

Question 7.2. If $\mathrm{char} k > 0$, is Definition 7.1 independent of the choice of k ?

(Said differently: does Definition 7.1 only depend on K , a function field in 2 variables over a finite field, and on A ?)

Question 7.3. Can one interpret $L(K, h^1(A), s)$, via a trace formula, as an “Euler” product of the form

$$L(C, j_* H_l^1(A), s) = \prod_{x \in C_{(0)}} L(k(x), i_x^* H_l^1(\mathcal{A}), s)$$

where \mathcal{A} is the Néron model of A over C ?

(It is not even clear that the right hand side converges!)

Let us now place ourselves in the situation of (0.2). Set $L(K, h^1(\Gamma), s) = L(K, h^1(J), s)$, and define similarly:

$$L(K, h^0(\Gamma), s) = L(k, h(C), s) \\ L(K, h^2(\Gamma), s) = L(k, h(C), s - 1) \\ L(K, h(\Gamma), s) = \prod_{i=0}^2 L(k, h^i(\Gamma), s).$$

Theorem 4.1 then gives the following analogue to Theorem 3:

Theorem 7.4. *One has*

$$\frac{L(k, h(S), s)}{L(K, h(\Gamma), s)} = L(k, D(-1), s) = L(k, D, s - 1). \quad \square$$

Question 7.5. The height pairing defined by Schneider in [Sch, p. 507]:

$$\mathcal{A}^0(C_s) \times A'(Kk_s) \rightarrow \text{Pic}(C_s)$$

induces a pairing

$$\mathcal{A}^0(C_s)/B(k_s) \times \text{LN}(A', Kk_s/k_s) \rightarrow \mathbf{Z}$$

because $B(k_s)$ and $B'(k_s)$ are divisible; moreover it presumably restricts to a pairing

$$(7.1) \quad B(k_s) \times \text{LN}(A', Kk_s/k_s) \rightarrow \text{Pic}^0(C_s).$$

One way to justify (7.1) would be to show that the functor $S \mapsto \Gamma(S, \mathcal{A} \times_k S)$ on k -schemes of finite type is representable by a k -group scheme of finite type with connected component B , and that Schneider's pairing emanates from a pairing of k -group schemes. Then (7.1) would induce a Galois-equivariant homomorphism

$$\text{LN}(A', Kk_s/k_s) \rightarrow \text{Hom}_{k_s}(B, J).$$

Can one use these pairings to describe the special values of $L(K, A, s)$?

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IMJ-PRG, CASE 247, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
E-mail address: bruno.kahn@imj-prg.fr